Math 155, *Lecture Notes- Bonds* **Name**____________

Section 8.2 Integration by Parts

Recall that the product rule for differentiation states that

$$\frac{d}{dx}(u(x)v(x)) = u(x)\frac{d}{dx}(v(x)) + v(x)\frac{d}{dx}(u(x))$$

or

$$u(x)\frac{d}{dx}(v(x)) = \frac{d}{dx}(u(x)v(x)) - v(x)\frac{d}{dx}(u(x))$$

Integration on both sides of this equation gives the following:

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx$$

Rewriting with *dv* and *du* differentials gives us

$$\int udv = uv - \int vdu$$

> **THEOREM 8.1 Integration by Parts**
>
> If u and v are functions of x and have continuous derivatives, then
>
> $$\int u\,dv = uv - \int v\,du.$$

Ex.1 Integrate: $\int \underset{u}{x}\underset{dv}{\sin(x)dx} = \int u\,dv$

$= (x)(-\cos(x)) - \int (-\cos(x))(dx)$

$= -x\cos(x) + \int \cos(x)\,dx$

$\boxed{= -x\cos(x) + \sin(x) + C}$ ✓

Let
$u = x$
$\frac{du}{dx} = 1$
$du = \frac{du}{dx}\cdot dx$
$du = 1\cdot dx$
$du = dx$

Let
$dv = \sin(x)dx$
$\frac{dv}{dx} = \sin(x)$
$\int\left(\frac{dv}{dx}\right)dx = \int \sin(x)\,dx$
$\int dv = -\cos(x) + C$
$v = -\cos(x) + C$
$v = -\cos(x)$

check: $\frac{d}{dx}[-x\cos(x) + \sin(x) + C]$

$= -\frac{d}{dx}[x\cos(x)] + \frac{d}{dx}[\sin(x)] + 0$

$= -\left[(x)\cdot\frac{d}{dx}(\cos(x)) + \cos(x)\cdot\frac{d}{dx}(x)\right] + \cos(x)$

$= -[x\cdot(-\sin(x)) + \cos(x)] + \cos(x)$

$= x\sin(x) - \cos(x) + \cos(x) = \underline{x\sin(x)}$ ✓

$$\int u\,dv = uv - \int v\,du$$

Ex.2 Evaluate: $\int \ln(3x)dx = \int 1 \cdot \ln(3x)dx$

$= \int u\,dv = uv - \int v\,du$

$= (\overset{u}{\ln(3x)})(\overset{v}{x}) - \int (\overset{v}{x}) \cdot (\overset{du}{\tfrac{1}{x}dx})$

$= x\ln(3x) - \int 1\,dx$

$$\boxed{= x\ln(3x) - x + C}$$

check: ?

Let
$u = \ln(3x)$
$\frac{du}{dx} = \frac{1}{3x} \cdot \frac{d}{dx}(3x)$
$\frac{du}{dx} = \frac{1}{3x} \cdot 3$
$\frac{du}{dx} = \frac{1}{x}$
$du = \frac{1}{x}dx$

Let
$dv = 1 \cdot dx$
$\frac{dv}{dx} = 1$
$\int \left(\frac{dv}{dx}\right)dx = \int 1\,dx$
$\int dv = x + C$
$v = x + C$
$v = x$

$$\int u\,dv = uv - \int v\,du$$

Ex.3 Evaluate: $\int x^2\cos(x)dx = \int u\,dv$

$= uv - \int v\,du$

$= (\overset{u}{x^2})(\overset{v}{\sin(x)}) - \int (\overset{v}{\sin(x)}) \cdot (\overset{du}{2x\,dx})$

$= x^2\sin(x) - 2 \cdot \underbrace{\int x\sin(x)\,dx}_{\text{Example \#1}}$

$= x^2\sin(x) - 2 \cdot [(x) \cdot (-\cos(x)) - \int (-\cos(x))(dx)]$

$= x^2\sin(x) - 2[-x\cos(x) + \int \cos(x)\,dx]$

$= x^2\sin(x) - 2[-x\cos(x) + \sin(x)] + C$

$$\boxed{= x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C}$$ ✓

check: ?

Let
$u = x^2$
$\frac{du}{dx} = 2x$
$du = 2x\,dx$

Let
$dv = \cos(x)\,dx$
$\frac{dv}{dx} = \cos(x)$
$\int dv = \int \cos(x)\,dx$
$v = \sin(x) + C$
$v = \sin(x)$

Let
$u = x$
$\frac{du}{dx} = 1$
$du = dx$

Let
$dv = \sin(x)dx$
$\frac{dv}{dx} = \sin(x)$
$\int dv = \int \sin(x)dx$
$v = -\cos(x) + C$
$v = -\cos(x)$

$$\int u\,dv = uv - \int v\,du$$

Ex.4 Evaluate: $\displaystyle\int_0^1 x\arcsin\left(x^2\right)dx$

$$= \left[(\underbrace{\arcsin(x^2)}_{u}\cdot(\underbrace{\tfrac{x^2}{2}}_{v})\right]_{x=0}^{x=1} - \int_{x=0}^{x=1}(\underbrace{\tfrac{x^2}{2}}_{v})\cdot\left(\underbrace{\frac{2x}{\sqrt{1-x^4}}dx}_{du}\right)$$

$$= \left[\frac{x^2}{2}\arcsin(x^2)\right]_0^1 - \int_0^1 \frac{x^3}{\sqrt{1-x^4}}dx$$

$$= \frac{(1)^2}{2}\arcsin(1^2) - (0) - \int_{w=1}^{w=0}\frac{x^3}{\sqrt{w}}\cdot\left(\frac{dw}{-4x^3}\right)$$

$$= \frac{1}{2}\arcsin(1) + \frac{1}{4}\cdot\int_{w=1}^{w=0} w^{-1/2}\,dw$$

$$= \frac{1}{2}\arcsin(1) + \frac{1}{4}\left[\frac{2}{1}w^{1/2}\right]_1^0$$

$$= \frac{1}{2}\arcsin(1) + \frac{1}{2}\left[(0)^{1/2} - (1)^{1/2}\right]$$

$$= \frac{1}{2}\arcsin(1) - \frac{1}{2}$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}$$

★ $\arcsin(1) = \frac{\pi}{2}$

$$\boxed{= \frac{\pi}{4} - \frac{1}{2}} \text{ or } \boxed{\frac{\pi - 2}{4}}$$

check on Ti-83?

Let

$u = \arcsin(x^2)$

$\frac{du}{dx} = \frac{1}{\sqrt{1-(x^2)^2}}\cdot\frac{d}{dx}[x^2]$

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^4}}\cdot 2x$

$\frac{du}{dx} = \frac{2x}{\sqrt{1-x^4}}$

$du = \frac{2x}{\sqrt{1-x^4}}dx$

$dv = x\,dx$

$\frac{dv}{dx} = x$

$\int dv = \int x\,dx$

$v = \frac{x^2}{2} + c$

$v = \frac{x^2}{2}$

Let $w = 1 - x^4$

$\frac{dw}{dx} = -4x^3$

$dw = \frac{dw}{dx}\cdot dx$

$dw = -4x^3dx$

$\frac{dw}{-4x^3} = dx$

$x = 0,$
$w = 1 - (0)^4$
$w = 1$ ✓

$x = 1,$
$w = 1 - (1)^4$
$w = 1 - 1$
$w = 0$ ✓

Let $\theta = \arcsin(1)$

$\sin(\theta) = \sin[\arcsin(1)]$

$\sin(\theta) = 1$

$\theta = \frac{\pi}{2}$

$\arcsin(1) = \frac{\pi}{2}$

$$\int u\,dv = uv - \int v\,du$$

Ex.5 Evaluate: $\int x^2e^{2x}dx = \int u\,dv$

$$= \overset{u}{(x^2)}\cdot\overset{v}{\left(\tfrac{1}{2}e^{2x}\right)} - \int \overset{v}{\left(\tfrac{1}{2}e^{2x}\right)}\cdot\overset{du}{(2x\,dx)}$$

$$= \tfrac{1}{2}x^2e^{2x} - \underbrace{\int xe^{2x}dx}_{\int udv = uv - \int vdu}$$

$$= \tfrac{1}{2}x^2e^{2x} - \left[(x)\cdot\left(\tfrac{1}{2}e^{2x}\right) - \int\left(\tfrac{1}{2}e^{2x}\right)(dx)\right]$$

$$= \frac{x^2}{2}e^{2x} - \left[\frac{x}{2}e^{2x} - \frac{1}{2}\int e^{2x}dx\right]$$

$$= \frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{1}{2}\int e^{2x}dx$$

$$= \frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{1}{2}\cdot\left[\frac{1}{2}e^{2x}\right] + C$$

$$\boxed{= \frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{1}{4}e^{2x} + C}$$

<u>check:</u> $\frac{d}{dx}\left[\frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{1}{4}e^{2x} + C\right]$

$$= \frac{1}{2}\cdot\frac{d}{dx}[x^2e^{2x}] - \frac{1}{2}\cdot\frac{d}{dx}[xe^{2x}] + \frac{1}{4}\cdot\frac{d}{dx}[e^{2x}] + 0$$

$$= \frac{1}{2}[(e^{2x})\cdot(2x) + (x^2)(e^{2x}\cdot 2)] - \frac{1}{2}[(e^{2x})(1) + (x)(e^{2x}\cdot 2)] + \frac{1}{4}[e^{2x}\cdot 2]$$

$$= \frac{1}{2}[2xe^{2x} + 2x^2e^{2x}] - \frac{1}{2}[e^{2x} + 2xe^{2x}] + \frac{1}{2}e^{2x}$$

$$= xe^{2x} + x^2e^{2x} - \frac{1}{2}e^{2x} - xe^{2x} + \frac{1}{2}e^{2x}$$

$$= x^2e^{2x}$$ ✓

Let $u = x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$dv = e^{2x}dx$

$\frac{dv}{dx} = e^{2x}$

$\int dv = \int e^{2x}dx$

$v = \int e^z\left(\frac{dz}{2}\right)$

Let $z = 2x$

$\frac{dz}{dx} = 2$

$\frac{dz}{2} = dx$

$v = \frac{1}{2}\int e^z dz$

$v = \frac{1}{2}\cdot e^z + C$

$v = \frac{1}{2}e^{2x}$

Let $u = x$

$\frac{du}{dx} = 1$

$du = dx$

Let

$dv = e^{2x}dx$

$\frac{dv}{dx} = e^{2x}$

$\int dv = \int e^{2x}dx$

$v = \frac{1}{2}e^{2x} + C$

$v = \frac{1}{2}e^{2x}$

$$\int u\,dv = uv - \int v\,du$$

Ex.6 Evaluate: $\int e^x \cos(2x)\,dx = \int u\,dv$

$= (\cos(2x))\cdot(e^x) - \int (e^x)(-2\sin(2x)\,dx)$

$= e^x\cos(2x) + 2\int \underline{e^x\sin(2x)\,dx}$

AGAIN?

$= e^x\cos(2x) + 2\cdot\left[(\sin(2x))(e^x) - \int (e^x)(2\cos(2x)\,dx)\right]$

$= e^x\cos(2x) + 2\left[e^x\sin(2x) - 2\int e^x\cos(2x)\,dx\right]$

$= e^x\cos(2x) + 2e^x\sin(2x) - 4\int \underline{e^x\cos(2x)\,dx}$

So, we have this again!!

$\int e^x\cos(2x)\,dx = e^x\cos(2x) + 2e^x\sin(2x) - 4\int e^x\cos(2x)\,dx$

$+4\int e^x\cos(2x)\,dx = \qquad +4\int e^x\cos(2x)\,dx$

$5\int e^x\cos(2x)\,dx = e^x\cos(2x) + 2e^x\sin(2x) + C$

$$\boxed{\int e^x\cos(2x)\,dx = \frac{1}{5}\left[e^x\cos(2x) + 2e^x\sin(2x)\right] + C}$$

Let $u = \cos(2x)$

$\frac{du}{dx} = -\sin(2x)\cdot 2$

$\frac{du}{dx} = -2\sin(2x)$

$du = -2\sin(2x)\,dx$

Let $dv = e^x\,dx$

$\frac{dv}{dx} = e^x$

$\int dv = \int e^x\,dx$

$v = e^x + C$

$v = e^x$

Let $u = \sin(2x)$

$\frac{du}{dx} = \cos(2x)\cdot 2$

$\frac{du}{dx} = 2\cos(2x)$

$du = 2\cos(2x)\,dx$

Let $dv = e^x\,dx$

$\frac{dv}{dx} = e^x$

$v = e^x$

check:?

Wow!

"Loop"

$$\int u\,dv = uv - \int v\,du$$

Ex.7 Evaluate: $\int_0^{\frac{\pi}{4}} x\sec^2(x)dx = \int_0^{\pi/4} u\,dv$

$= \left[\overset{u}{(x)}\cdot\overset{v}{(\tan(x))}\right]_0^{\pi/4} - \int_0^{\pi/4} \overset{v}{(\tan(x))}\cdot\overset{du}{(dx)}$

$= \left[x\tan(x)\right]_0^{\pi/4} - \int_0^{\pi/4}\tan(x)dx$

$= \left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right) - 0 - \left[-\ln|\cos(x)|\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{4}\cdot 1 + \left[\ln|\cos(x)|\right]_0^{\pi/4}$

$= \frac{\pi}{4} + \left(\ln\left|\cos\left(\frac{\pi}{4}\right)\right| - \ln|\cos(0)|\right)$

$= \frac{\pi}{4} + \ln\left|\frac{\sqrt{2}}{2}\right| - \ln|1|$

$= \frac{\pi}{4} + \ln\left(\frac{\sqrt{2}}{2}\right) - 0$

$\boxed{= \frac{\pi}{4} + \ln\left(\frac{\sqrt{2}}{2}\right)}$ or

$= \frac{\pi}{4} + \ln(\sqrt{2}) - \ln(2)$

$= \frac{\pi}{4} + \frac{1}{2}\ln(2) - \ln(2)$

$= \boxed{\frac{\pi}{4} - \frac{1}{2}\ln(2)}$

★ $\ln(\sqrt{2}) = \ln(2^{\frac{1}{2}})$
$= \frac{1}{2}\ln(2)$

Let $u = x$
$\frac{du}{dx} = 1$
$du = dx$

$dv = \sec^2(x)dx$
$\frac{dv}{dx} = \sec^2(x)$
$\int dv = \int \sec^2(x)dx$
$v = \tan(x) + c$
$v = \tan(x)$

Tabular Method

$$\int u\,dv = uv - \int v\,du$$

Ex.8 Evaluate: $\int x^3 e^{-2x}\,dx$

$= +x^3\cdot\left(-\frac{1}{2}e^{-2x}\right) - 3x^2\cdot\left(\frac{1}{4}e^{-2x}\right) + 6x\cdot\left(-\frac{1}{8}e^{-2x}\right) - 6\cdot\left(\frac{1}{16}e^{-2x}\right) + C$

$$\boxed{= \frac{-x^3}{2}e^{-2x} - \frac{3x^2}{4}e^{-2x} - \frac{3x}{4}e^{-2x} - \frac{3}{8}e^{-2x} + C}$$

OR

$$\boxed{= -e^{-2x}\left(\frac{x^3}{2} + \frac{3x^2}{4} + \frac{3x}{4} + \frac{3}{8}\right) + C}\ \checkmark$$

check: ?

sign	Differentiate u	Integrate dv
$+$	x^3	e^{-2x}
$-$	$3x^2$	$-\frac{1}{2}e^{-2x}$
$+$	$6x$	$\frac{1}{4}e^{-2x}$
$-$	6	$-\frac{1}{8}e^{-2x}$
$+$	0	$\frac{1}{16}e^{-2x}$

$\int e^{-2x}dx$, Let $z = -2x$, $\frac{dz}{dx} = -2$, $\frac{dz}{-2} = dx$

$= \int e^{z}\left(\frac{dz}{-2}\right)$

$= -\frac{1}{2}\int e^{z}dz$

$= -\frac{1}{2}e^{z} + C$

$= -\frac{1}{2}e^{-2x}$

$\int \left(-\frac{1}{2}e^{-2x}\right)dx$

$= -\frac{1}{2}\int e^{-2x}dx$

$= -\frac{1}{2}\cdot\left[-\frac{1}{2}e^{-2x}\right] + C$

$= \frac{1}{4}e^{-2x} + C = \frac{1}{4}e^{-2x}$